Principles of Macroeconomics: A Production Economy Part 2

Class 5

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Overview

- Announcements:
  - You should be able to do LC 7 and GH 7
  - You should be able to do most of LC 9 and GH 9 (due September 12th at 11:59pm)
- Topics:
  - Recap Growth Model
  - Development Accounting
- Readings:
  - Chapters 9.3-9.4

Production Function Recap

- We used the Cobb-Douglas Production Function:

$$Y = AK^{\alpha}L^{1-\alpha}$$

- The α and $1-\alpha$ structure in the exponents gives constant returns to scale
  - 2x inputs $\longrightarrow$ 2x output
- Optimization gave:
  - MPL: $\frac{\partial Y}{\partial L} \geq 0$
  - MPK: $\frac{\partial Y}{\partial L} \geq 0$
  - But with diminishing marginal products: as L/K increase, MPL/MPK decrease

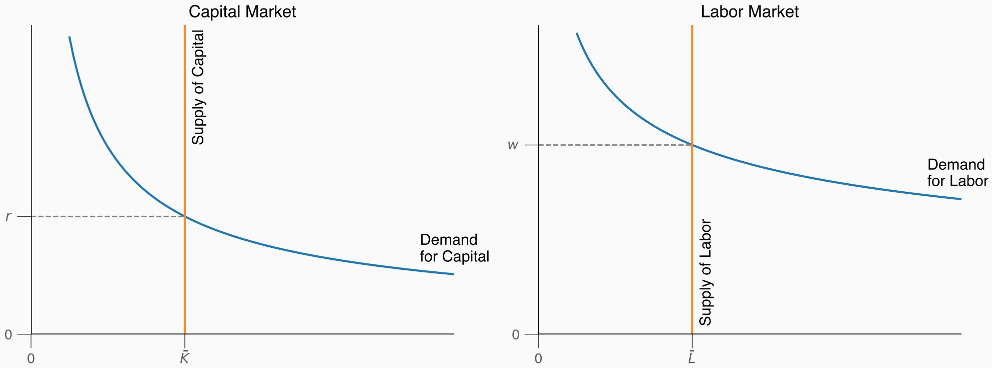
Model Overview

1. The Production Function ✓
2. Factor Supply ✓
    - Fixed endowment of labor: $L = \bar{L}$
    - Fixed endowment of capital: $K = \bar{K}$
3. Producer behavior ✓
    - Competitive producers maximize profits
    - These producers demand capital and labor
4. Equilibrium
    - Set supply $=$ demand
    - Solve for prices and output

Equilibrium

- Producer decisions are in fact *demand* curves
  - $\downarrow w \longrightarrow \uparrow L$
  - $\downarrow r \longrightarrow \uparrow K$
- Producers have downward-sloping demand for K and L
- We fixed the supply of K and L
- We now equate supply and demand

Graphical Equilibrium



Suppose that $A = 2$, $\alpha = 1/3$, $\bar{K} = 100$, $\bar{L} = 50$, and $p = 1$. Let the production function be a constant returns-to-scale Cobb Douglas function

(1) Compute Y

(2) Derive firm demand for capital and labor (no need to take derivatives)

(3) Find the factor prices r and w

(4) Verify that capital expenditure relative to output is α. Verify that labor expenditure relative to output is $1 - \alpha$

(5) Suppose A increases by 10%. By what percentage do Y, r, and w change?

Solutions

(1) $Y = 2 \times 100^{1/3} \times 50^{2/3} \approx 126$

(2) The profit function is: $\Pi = pY - wL - rK$. Using the derivatives with respect to K and L from class, we get that MB = MC:

$$\frac{\partial \Pi}{\partial K} : r = \alpha A K^{\alpha - 1} L^{1-\alpha}$$
$$\frac{\partial \Pi}{\partial L} : w = (1-\alpha) A K^{\alpha} L^{-\alpha}$$

(3) Plug-in our specific calibration:

$$r = \frac{1}{3} \times 2 \times 100^{-2/3} \times 50^{2/3} \approx 0.42$$
$$w = \frac{2}{3} \times 2 \times 100^{1/3} \times 50^{-1/3} \approx 1.68$$

(4) Plug-in our solutions:

$$\alpha = \frac{rK}{pY} \qquad\qquad 1-\alpha = \frac{wL}{pY}$$

$$\frac{1}{3} = \frac{0.42 \times 100}{126} \qquad\qquad \frac{2}{3} = \frac{1.68 \times 50}{126}$$

$$\frac{1}{3} = \frac{1}{3}\checkmark \qquad\qquad \frac{2}{3} = \frac{2}{3}\checkmark$$

(5) For Y: $100 \times \frac{(1.1)A_0K^\alpha L^{1-\alpha} - A_0K^\alpha L^{1-\alpha}}{A_0K^\alpha L^{1-\alpha}} = 10\%$

For r: $100 \times \frac{\alpha(1.1)A_0K^{\alpha-1}L^{1-\alpha} - A_0K^{\alpha-1}L^{1-\alpha}}{A_0K^{\alpha-1}L^{1-\alpha}} = 10\%$

For w: $100 \times \frac{(1-\alpha)(1.1)A_0K^\alpha L^{-\alpha} - (1-\alpha)A_0K^\alpha L^{-\alpha}}{(1-\alpha)A_0K^\alpha L^{-\alpha}} = 10\%$

RGDP per Capita

- Remember that RGDP per capita (or per person) is our measure of the standard of living
- To proxy this, we will assume that workers = population
- So divide output by labor:

$$Y = AK^{\alpha}L^{1-\alpha}$$
$$\frac{Y}{L} = A\frac{K}{L}^{\alpha}$$

- If we denote lower-case letters as "x per capita", then:

$$y = Ak^{\alpha}$$

Productivity and Capital Intensity

- $y = Ak^{\alpha}$
  - A: productivity
    - More productive economies are richer
  - k: capital per person
    - Workers equipped with more capital produce more
    - But there's diminishing returns – doubling capital per person does not double output per person

Evaluating our Model

- This model has a lot of assumptions...
  - Production is Cobb-Douglas
  - Constant returns to scale
  - Perfectly competitive factor markets
  - Perfectly inelastic supply of factors
  - Closed economy
  - Etc, etc
- But is it useful still? Can it make sense of the data
  - A model must make simplifying assumptions – a good model uses assumptions that:
    - Clarify the logic
    - Can be checked by data (either directly or indirectly)
    - Is useful even when the assumptions aren't exactly met

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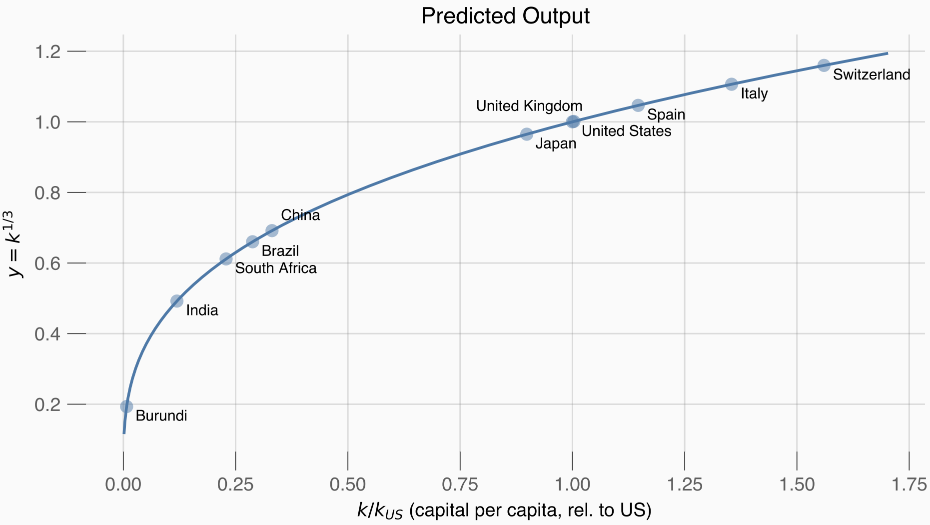
Development Accounting

- To test the model, we will study cross-country income differences using the production function:

$$y = Ak^{\frac{1}{3}}$$

- Data:
  - $y \equiv$ real GDP per capita
  - $k \equiv$ capital per capita
  - Use data from the World Bank and Penn World Tables
- What about A?
  (1) Assume A is the same across all countries, set $A = 1$
  (2) Let A differ, use model to back-out A
  - We will use (1) first

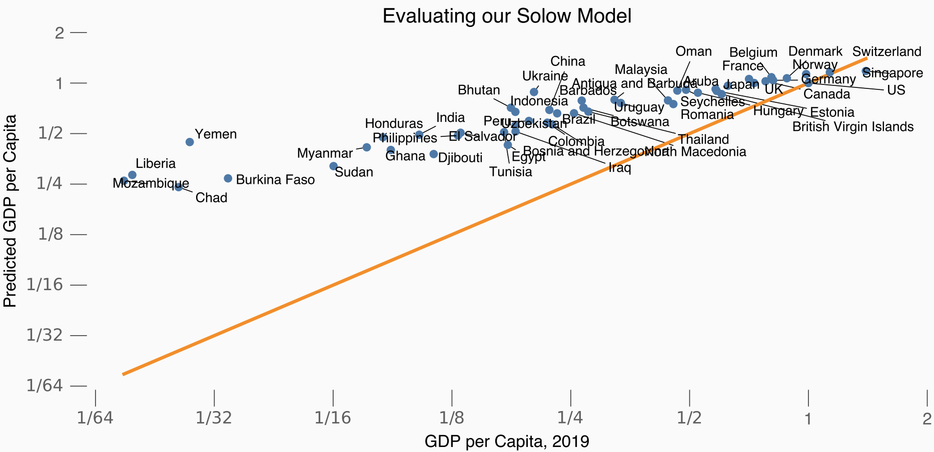
Predicted Income Differences



Compare to Data

Country	Observed capital per person	Predicted GDP per Capita	Observed GDP per Capita
United States	1	1	1
Burundi	0.007	0.193	0.012
Brazil	0.288	0.66	0.231
Switzerland	1.56	1.16	1.133
China	0.331	0.692	0.221
Spain	1.146	1.047	0.652
United Kingdom	1.003	1.001	0.729
India	0.119	0.492	0.103
Italy	1.355	1.106	0.653
Japan	0.898	0.965	0.625
South Africa	0.229	0.612	0.202

Graphically



So our model thinks countries should be richer than they really are!

Marginal Product of Capital

- Recall the production function: low $k \longrightarrow$ high MPK

$$MPK = p\left[\frac{\partial Y}{\partial K}\right] = \frac{p}{3}Ak^{-2/3}$$

- So capital scarce countries should have high MPK. Is this what we see?
- Puzzle 1: Economists have calculated MPK for many countries. We find:
  - Rich countries: 8.4%
  - Poor countries: 6.9%
- Puzzle 2: Why isn't capital going to poor countries?
  - Low $k \longrightarrow$ high returns to investment
  - So k should go to poor countries
  - Not so (Lucas Paradox)

Evaluation

- Success: differences in k explains some of the differences in y across countries
- Failure: differences in k don't explain most of the differences in y
  - Countries are poorer than expected
  - Returns to capital are actually lower in poor countries
- What should we do? What key assumption did we make?

Development Accounting Try 2

- A is also defined as TFP – Total Factor Productivity
- It is essentially a parameter that tells us how effective a country is at turning K and L into Y
- Revisit the production function:

$$y = Ak^{1/3}$$

- If k doesn't explain the differences in y, maybe A can?

Problem: How do we measure A?

- There is no "TFP" object out there
- So what do we do? We measure TFP as a residual:

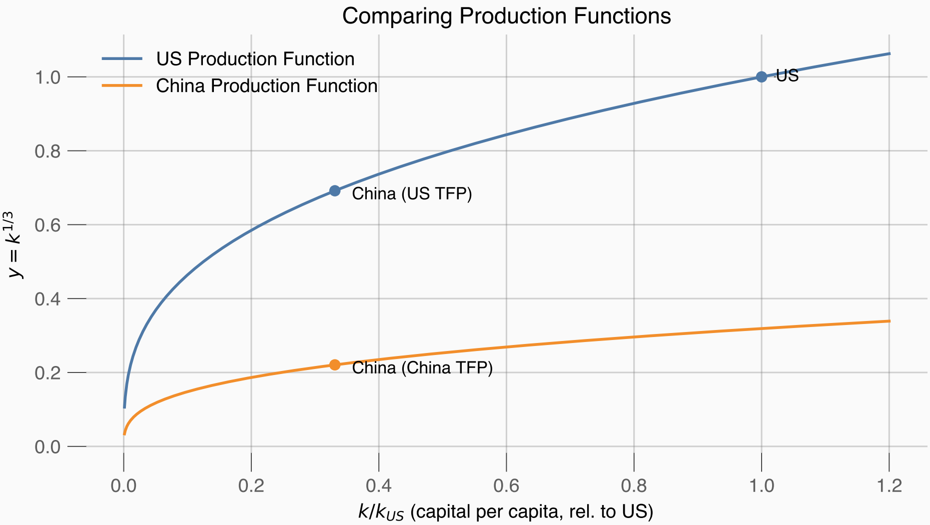
$$A = \frac{y}{k^{1/3}}$$

- Essentially, A becomes the part of productivity that we don't understand

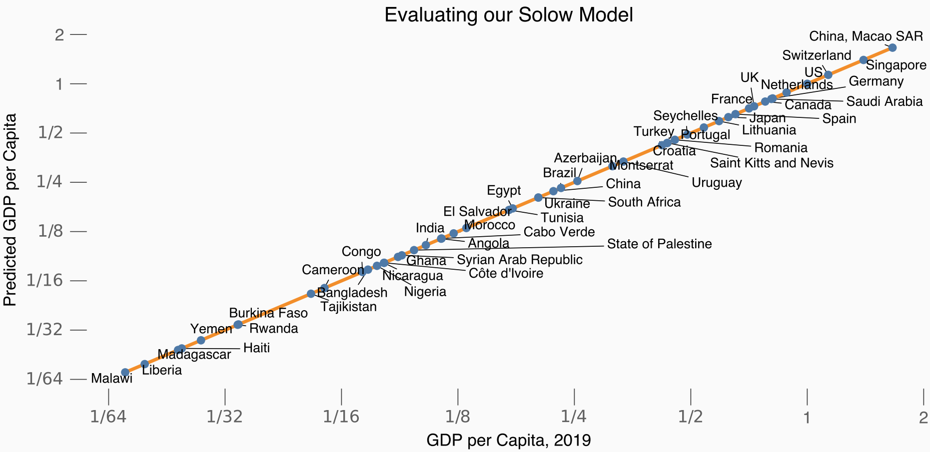
Implied TFP

Country	Observed GDP per person	Observed capital per person	Implied TFP
United States	1	1	1
Brazil	0.231	0.288	0.35
Burundi	0.012	0.007	0.061
China	0.221	0.331	0.319
India	0.103	0.119	0.21
Italy	0.653	1.355	0.591
Japan	0.625	0.898	0.648
South Africa	0.202	0.229	0.33
Spain	0.652	1.146	0.623
Switzerland	1.133	1.56	0.977
United Kingdom	0.729	1.003	0.728

US vs. China


Comparing Production Functions
US Production Function
China Production Function
US
China (US TFP)
China (China TFP)
$y = k^{1/3}$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
k/k_{US} (capital per capita, rel. to US)

Graphically



Capital vs. TFP

- Is k or A more important in this model?
- If we compare the five richest and five poorest economies:

$$\underbrace{\frac{y_{rich}}{y_{poor}}}_{126} \approx \underbrace{\frac{A_{rich}}{A_{poor}}}_{23} \underbrace{\left(\frac{k_{rich}}{k_{poor}}\right)^{1/3}}_{5}$$

- TFP accounts for roughly 80% of the income differences
- k accounts for only 20%
- Is this a good model?

Practice Problems

Suppose output per worker and capital per worker are as given in the chart below.

Country	y	k
U.S.	1.00	1.00
Country P	0.15	0.08
Country R	1.20	1.40

(1) Compute TFP for country P and country R

(2) Compute the MPK for countries P and R relative to the US. Where should capital flow?

(3) Decompose the income gap between P and the US into contributions from TFP and capital.

(4) Suppose A_P jumps to 1. What is the percentage increase in y_P?

(5) Find k_P^* such that $\text{MPK}_P = \text{MPK}_{US}$, then compute y_P^* and the percent rise vs. today.

(6) What is a friction that might explain large income gaps but only modest MPK gaps? Does this friction primarily impact the TFP gap or the capital gap?

(1) We use the Cobb-Douglas form we assumed for the production function:

$$A_j = y_j k_j^{-1/3}$$

Plugging in the numbers from the table:

$$\begin{aligned} A_P &= y_P k_p^{-1/3} \\ &= 0.15 \times 0.08^{-1/3} \\ &\approx 0.35 \end{aligned} \qquad \begin{aligned} A_R &= y_R k_R^{-1/3} \\ &= 1.20 \times 1.40^{-1/3} \\ &\approx 1.07 \end{aligned}$$

(2) The MPK is $\frac{1}{3}A_j k_j^{-2/3}$. Apply this to our problem:

$$\begin{aligned} \text{MPK}_P &= \frac{1}{3}A_P k_P^{-2/3} \\ &= \frac{1}{3} \times 0.35 \times (0.08)^{-2/3} \\ &\approx 0.625 \end{aligned}$$

$$\begin{aligned} \text{MPK}_R &= \frac{1}{3}A_R k_R^{-2/3} \\ &= \frac{1}{3} \times 1.07 \times (1.40)^{-2/3} \\ &\approx 0.286 \end{aligned}$$

We also need MPK for the US:

$$\text{MPK}_{US} = \frac{1}{3}A_{US} k_{US}^{-2/3} = \frac{1}{3}$$

Now calculate the ratios:

$$\begin{aligned} \frac{\text{MPK}_P}{\text{MPK}_{US}} &= \frac{0.625}{0.33} \\ &\approx 1.88 \end{aligned}$$

$$\begin{aligned} \frac{\text{MPK}_R}{\text{MPK}_{US}} &= \frac{0.286}{0.33} \\ &\approx 0.86 \end{aligned}$$

(3) We use the income ratio formula: $\frac{y_j}{y_i} = \frac{A_j}{A_i}\left(\frac{k_j}{k_i}\right)^{\frac{1}{3}}$. So:

$$\frac{y_{US}}{y_P} = \frac{A_{US}}{A_P}\left(\frac{k_{US}}{k_P}\right)^{1/3}$$

$$\frac{1}{0.15} = \frac{1}{0.35}\left(\frac{1}{0.08}\right)^{1/3}$$

$$6.67 \approx 2.87 \times 2.32$$

$$\ln(6.67) \approx \ln(2.87) + \ln(2.32)$$

Decompose in percent terms:

$$\text{A}: \quad 100 \times \frac{\ln(2.87)}{\ln(6.67)} \approx 55.6\% \qquad \text{k}: \quad 100 \times \frac{\ln(2.32)}{\ln(6.67)} \approx 44.4\%$$

(4) Calculate the new GDP per worker:

$$y_P' = 1 \times 0.08^{1/3} \approx 0.43$$

In percent gain:

$$g_y = 100 \times \frac{0.43 - 0.15}{0.15} \approx 187\%$$

(5) Set MPK_P equal to MPK_{US}:

$$\begin{aligned}
\alpha A_P \left(k_P^*\right)^{\alpha-1} &= A_{US} k_{US}^{\alpha-1} \\
k_p^* &= \left(\frac{A_{US}}{A_P}\right)^{\frac{1}{\alpha-1}} k_{US} \\
&= \left(\frac{1}{0.35}\right)^{-\frac{3}{2}} \\
&\approx 0.205
\end{aligned}$$

Now find y_P^*:

$$\begin{aligned}
y_P^* &= A_P \left(k_p^*\right)^{1/3} \\
&= 0.35 \times (0.205)^{1/3} \\
&\approx 0.205
\end{aligned}$$

In percent growth:

$$g_Y = 100 \times \frac{0.205 - 0.15}{0.15} \approx 36.9\%$$

(6) *Institutional/technology adoption barriers* (lowers A): weak enforcement, corruption, managerial gaps; extra k yields limited gains

Risk and intermediation costs (keeps k low): sovereign/currency risk and shallow finance raise required returns, limiting capital inflows even with high gross MPK – so MPK might be high, but investors won't invest in the country

Summary

- Simplified production economy, applied to data, predicts that TFP matters more than capital per worker
- Natural question: how do we increase TFP?
- More on this on Thursday